Inexact sequential injective algorithm for weakly univalent vector equation and its application to regularized smoothing Newton algorithm for mixed second-order cone complementarity problems

Shunsuke Hayashi*

September 20, 2018

Abstract It is known that the complementarity problems and the variational inequality problems are reformulated equivalently as a vector equation by using the natural residual or Fischer-Burmeister function. In this paper, we first propose an inexact sequential injective algorithm (ISIA) for a vector equation, and show the global convergence under weak univalence assumption. Roughly speaking, the ISIA generates the sequence of inexact solutions of approximate vector equations, which consist of the injectives converging to the original vector-valued function. Although the ISIA is simple and conceptual, it can be a prototype to many other algorithms such as a smoothing Newton algorithm, semismooth Newton algorithm, etc. Next, we apply the ISIA prototype to the regularized smoothing Newton algorithm (ReSNA) for mixed second-order cone complementarity problems (MSOCCPs). Exploiting the ISIA convergence scheme, we prove that the ReSNA is globally convergent under Cartesian P_0 assumption.

1 Introduction

In this paper, we first focus on the following vector equation (VE):

$$H(z) = 0, \tag{1.1}$$

where $H : \mathcal{D} \to \mathbb{R}^n$ is continuous over the domain $\mathcal{D} \subseteq \mathbb{R}^n$, but need not be differentiable.

Many classes of problems can be cast as VE (1.1). For example, fixed point problem "find $z \in \mathbb{R}^n$ such that $z = F(z)$" reduces to VE (1.1) by letting $H(z) := z - F(z)$. Nonlinear complementarity problem (NCP) is to find a vector $z \in \mathbb{R}^n$ such that $z \geq 0$, $F(z) \geq 0$ and $z^\top F(z) = 0$, for a given function $F : \mathbb{R}^n \to \mathbb{R}^n$. This problem also reduces to VE (1.1) by letting $H(z) := \min(z, F(z))$ or $H(z) := \left(\sqrt{z_i^2 + F_i(z)^2} - z_i - F_i(z)\right)_{i=1}^n$, where the latter one is well-known Fischer-Burmeister function. By using Euclidean Jordan algebra [7, 8], second-order cone complementarity problem (SOCCP) [2, 3, 4, 8, 11, 15, 16] and symmetric cone complementarity problem (SCCP) [1, 5, 14, 19] can be also reformulated as VE (1.1). Moreover, variational inequality problem (VIP) [6, 10] is also cast as VE (1.1) by using the Euclidean projection onto the convex set composed in the VIP.

A function $H : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called *weakly univalent* if it is continuous and there exists a sequence of continuous injective functions converging to H uniformly on any bounded subset of $\mathcal{D}$. (The detailed definition is given in Section 2.) Gowda and Sznajder [9] focused on this property and

*Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan (`s_hayashi@plan.civil.tohoku.ac.jp`)

analyzed the connectedness of solution set of linear complementarity problems (LCP). Qi and Sun [18] used this property to discuss the level-boundedness of merit function for complementarity problems. Hayashi, Yamashita and Fukushima [11] proposed regularized smoothing Newton method with inexact inner step calculation for solving SOCCP. They also used the weak univalence property to prove the global convergence.

In this paper, we propose the inexact sequential injective algorithm (ISIA) for solving VE (1.1) under the assumption that function H is weakly univalent. Moreover, we prove that the ISIA has global convergence property. In fact, the ISIA is quite conceptual since it does not contain explicit calculation process. The proof technique is based on the existing analyses on smoothing Newton methods. However, we can give the following advantages of considering and analyzing the ISIA.

- The ISIA can be a prototype to many other algorithms. (e.g., smoothing (and regularized) Newton type algorithm for complementarity problems)
- The global convergence of any algorithms can be discussed comprehensively if they meet the ISIA prototype.

In other words, *the ISIA reveals the essence of global convergence more clearly* than the existing smoothing Newton algorithms.

In the latter part of the paper, we focus on the regularized smoothing Newton algorithm (ReSNA) for mixed SOCCPs and apply the ISIA prototype to show the global convergence. Here, the word "mixed" implies that not only the SOC complementarity but also the equality conditions are contained. Although the ReSNA is proposed in [11] originally, the authors considered "non-mixed" SOCCP only, and proved the convergence under "monotonicity" assumption. On the other hand, we handle the "mixed" SOCCP and show the convergence under the *Cartesian* P_0 assumption, which is strictly weaker than the monotonicity. It is known that the KKT conditions of a general SOCP and some kinds of robust Nash equilibrium problems [12, 17] are reformulated as mixed SOCCPs. Therefore, from the viewpoint of applications, it is more convenient to deal with the mixed SOCCP rather than the non-mixed SOCCP only.

Throughout the paper, we use the following notations. $\mathbb{R}^n_+$ denotes the n-dimensional nonnegative orthant. $H^{-1}(v)$ denotes the inverse set-valued mapping of H, i.e., $\{z \mid H(z) = v\}$. Thus $H^{-1}(0)$ denotes the solution set of VE (1.1).

2 Inexact sequential injective algorithm for weakly univalent equation

In this section, we propose the inexact sequential injective algorithm for solving VE (1.1). For the sake of simplicity, we assume $\mathcal{D} = \mathbb{R}^n$ hereafter. In case of $\mathcal{D} \subsetneq \mathbb{R}^n$, the subsequent discussions can be extended in a direct manner. Now, we first introduce the weak univalence property for vector-valued function.

Definition 2.1 (weak univalence) *Function $H : \mathbb{R}^n \to \mathbb{R}^n$ is said to be weakly univalent if it is continuous and there exists a sequence of continuous and injective functions $\{\tilde{H}_k\}$ converging to H uniformly*[*1] *over any bounded subset of $\mathbb{R}^n$.*

[*1] We say that the sequence of functions $\{\tilde{H}_k\}$ converges to H uniformly over the bounded set Ω, if $\sup\{\|\tilde{H}_k(w) - H(w)\| : w \in \Omega\}$ converges to 0 as $k \to \infty$.

We can easily see that any weakly univalent function is continuous. Moreover, any P_0 function or monotone function is known to be weakly univalent. For VE (1.1), we assume the followings.

Assumption A *$H : \mathbb{R}^n \to \mathbb{R}^n$ in VE (1.1) satisfies the following conditions*:

(i) *H is weakly univalent*;

(ii) *The solution set $H^{-1}(0)$ is nonempty and bounded.*

We note that this assumption depends only on the problem itself, and does not depend on the applied algorithm or generated sequence. Needless to say, assumption (ii) is satisfied when VE (1.1) has a unique solution.

Now, we provide the inexact sequential injective algorithm.

Algorithm 1 (Inexact sequential injective algorithm: ISIA)

Step 0 *Choose the function and scalar sequences $\{\tilde{H}_k\}$ and $\{\beta_k\}$ such that*

(i) *$\beta_k \geq 0$ for all k, and $\lim_{k\to\infty} \beta_k = 0$;*

(ii) *For each k, $\tilde{H}_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and injective, and $\{\tilde{H}_k\}$ converges to H uniformly over any bounded set.*

Choose w^0 and set $k := 0$.

Step 1 *If $\|H(w^k)\| = 0$, then terminate and output w^k as a solution. Otherwise, go to Step 2.*

Step 2 *Find a vector $w^{k+1} \in \mathbb{R}^n$ such that*

$$\|\tilde{H}_k(w^{k+1})\| \leq \beta_k. \tag{2.1}$$

Step 3 *Set $k := k + 1$. Go back to Step 1.*

In Step 0, the sequence $\{\beta_k\}$ or $\{\tilde{H}_k\}$ need not be determined explicitly beforehand. It is sufficient if $\{\beta_k\}$ and $\{\tilde{H}_k\}$ satisfy conditions (i) and (ii) eventually through the iterations. To obtain w^{k+1} in Step 2, we may use any suitable unconstrained minimization technique or Newton type approach. In order for the ISIA to be well-defined, there must exist w^{k+1} satisfying (2.1).

Assumption B *For the functions $\{\tilde{H}_k\}$ and parameters $\{\beta_k\}$ used in Algorithm 1, $\{w \mid \|\tilde{H}_k(w)\| \leq \beta_k\}$ is nonempty for all k.*

Notice that Assumption B holds when $\tilde{H}_k^{-1}(0) \neq \emptyset$ or $\tilde{H}_k$ is a one-to-one mapping. Moreover, due to Assumption A(ii), Assumption B must hold for all k sufficiently large.

Now, we are to show the global convergence of the algorithm. To this end, we introduce the following lemma, which indicates a property that the weakly univalent functions possess.

Lemma 2.1 [6, Cor. 3.6.5] *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a weakly univalent function such that the inverse image $H^{-1}(0)$ is nonempty and compact. Then, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the following statement*[*2] *holds*:

[*2] Here, $\mathrm{cl}\,(\cdot)$ and $B(0, \varepsilon)$ denote the closure and the open ball with radius $\varepsilon > 0$, i.e., $B(0, \varepsilon) := \{w \in \mathbb{R}^n \mid \|w\| < \varepsilon\}$, respectively.

If a function $G : \mathbb{R}^n \to \mathbb{R}^n$ is weakly univalent and

$$\sup\left\{\|G(w) - H(w)\| : w \in \mathrm{cl}(H^{-1}(0) + B(0,\varepsilon))\right\} \leq \delta,$$

then $G^{-1}(0)$ is connected and $\emptyset \neq G^{-1}(0) \subseteq H^{-1}(0) + B(0,\varepsilon)$.

By using this lemma, we establish the global convergence of the ISIA.

Theorem 2.1 *Suppose that Assumptions A and B holds. Let $\{w^k\}$ be the sequence generated by Algorithm 1. Then, $\{w^k\}$ is bounded, and any accumulation point solves VE(1.1).*

proof. We first show the boundedness of $\{w^k\}$. Let $\varepsilon > 0$ be fixed arbitrarily. Then we have a positive number $\delta = \delta(\varepsilon) > 0$ satisfying Lemma 2.1. Moreover, by Assumption A(ii), $\Omega_\varepsilon := H^{-1}(0)+B(0,\varepsilon)$ is nonempty and bounded. Now, define $\tilde{G}_k : \mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{G}_k(w) := \tilde{H}_k(w)-\tilde{H}_k(w^{k+1})$ for each k. Then $\tilde{G}_k$ is weakly univalent since it is continuous and injective. Moreover, there exists a $\overline{k}$ such that the following formulas hold for any $k \geq \overline{k}$ and $w \in \mathrm{cl}\,\Omega_\varepsilon$:

$$\begin{aligned}\|\tilde{G}_k(w) - H(w)\| &= \|\tilde{H}_k(w) - \tilde{H}_k(w^{k+1}) - H(w)\| \\ &\leq \|\tilde{H}_k(w) - H(w)\| + \|\tilde{H}_k(w^{k+1})\| \\ &\leq \sup\left\{\|\tilde{H}_k(w') - H(w')\| : w' \in \mathrm{cl}\,\Omega_\varepsilon\right\} + \beta_k \\ &\leq \delta,\end{aligned}$$

where the first inequality is due to the triangular inequality, the second inequality holds from $w \in \mathrm{cl}\,\Omega_\varepsilon$ and (2.1), and the last inequality follows since β_k converges to 0 and $\{\tilde{H}_k\}$ converges to H uniformly over the compact set $\mathrm{cl}\,\Omega_\varepsilon$. Thus, by Lemma 2.1 with $G := \tilde{G}_k$, we have

$$\emptyset \neq \tilde{G}_k^{-1}(0) \subseteq \Omega_\varepsilon$$

for all $k \geq \overline{k}$. Since $w^{k+1} \in \tilde{G}_k^{-1}(0)$ and Ω_ε is bounded, we have the boundedness of $\{w^k\}$.

Next we show the latter part. Since $\{w^k\}$ is bounded, there exists a bounded set W satisfying $\{w^k\} \subseteq W$. By the triangular inequality and Step 2 of Algorithm 1, we have

$$\begin{aligned}\|H(w^{k+1})\| &\leq \|\tilde{H}_k(w^{k+1}) - H(w^{k+1})\| + \|\tilde{H}_k(w^{k+1})\| \\ &\leq \sup\left\{\|\tilde{H}_k(w') - H(w')\| : w' \in W\right\} + \beta_k.\end{aligned}$$

Thus we have $\|H(w^{k+1})\| \to 0$. Since H is continuous, arbitrary accumulation point w^* of $\{w^k\}$ satisfies $H(w^*) = 0$. ∎

We readily have the following corollary, which can be applied to the smoothing Newton type algorithms more directly.

Corollary 2.1 *Let $\tilde{H}_\nu : \mathbb{R}^n \to \mathbb{R}^n$ be a function with a vector parameter $\nu \in \mathbb{R}^\ell_+$ such that*

(i) *for any fixed $\nu > 0$, $\tilde{H}_\nu$ is continuously differentiable over $\mathbb{R}^n$,*

(ii) *$\nabla\tilde{H}_\nu(w)$ is nonsingular for any $\nu > 0$ and $w \in \mathbb{R}^n$,*

(iii) *$\tilde{H}_\nu$ converges to H uniformly over an arbitrary bounded set $\Omega \subset \mathbb{R}^n$ as $\nu \searrow 0$.*

Let $\{\nu_k\} \subseteq \mathbb{R}^\ell_+$ be an arbitrary sequence such that $\nu_k > 0$ for any k, and $\nu_k \searrow 0$ as $k \to \infty$. Let $\{w^k\}$ the sequence generated by Algorithm 1 with $\tilde{H}_k := H_{\nu_k}$. Suppose that Assumptions A and B hold. Then, $\{w^k\}$ is bounded, and any accumulation point solves VE (1.1).

proof. By (i) and (ii), the function H_{ν_k} is continuous and injective for each k. Thus, by (iii) and Theorem 2.1, we have the corollary. ∎

3 Regularized smoothing Newton algorithm for mixed second-order cone complementarity problem

In the previous section, we have studied the global convergence of ISIA. Our next step is to apply the ISIA prototype to the ReSNA for mixed second-order cone complementarity problems (MSOCCPs). In this section, we first review the background of MSOCCP briefly, and then extend the ReSNA for non-mixed SOCCP to MSOCCP in a straightforward manner. The actual convergence analysis will be provided in the next section.

3.1 Mixed second-order cone complementarity problem

The MSOCCP is formulated as follows:

$$\text{(MSOCCP)} \qquad \begin{array}{rl} \text{Find} & (x, y, p) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \\ \text{such that} & x \in \mathcal{K},\ y \in \mathcal{K},\ x^\top y = 0, \\ & y = F_1(x, p),\ F_2(x, p) = 0, \end{array} \tag{3.1}$$

where $F_1 : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^n$ and $F_2 : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^\ell$ are given continuously differentiable functions, and $\mathcal{K}$ is a Cartesian product of several second-order cones (SOCs), i.e.,

$$\mathcal{K} := \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_m} \tag{3.2}$$

with $n_1 + n_2 + \cdots + n_m = n$ and

$$\mathcal{K}^{n_i} := \begin{cases} \{z \in \mathbb{R} \mid z \ge 0\} & (n_i = 1) \\ \left\{ z \in \mathbb{R}^{n_i} \,\middle|\, z_1 \ge \sqrt{z_2^2 + \cdots + z_{n_i}^2} \right\} & (n_i \ge 2). \end{cases}$$

The MSOCCP involves many kinds of optimization/equilibrium problems as special cases. When $n_1 = n_2 = \cdots = n_m$ (i.e., $\mathcal{K} = \mathcal{K}^1 \times \mathcal{K}^1 \times \cdots \times \mathcal{K}^1 = \mathbb{R}^n_+$), MSOCCP (3.1) reduces to nonlinear complementarity problem (NCP) or mixed complementarity problem (MCP). On the other hand, for the nonlinear second-order cone programming problem (NSOCP)

$$\begin{array}{rl} \text{Minimize} & \theta(z) \\ \text{subject to} & G(z) \in \mathcal{K},\ \ H(z) = 0 \end{array} \tag{3.3}$$

with $\theta : \mathbb{R}^{\ell_1} \to \mathbb{R}$, $G : \mathbb{R}^{\ell_1} \to \mathbb{R}^n$ and $H : \mathbb{R}^{\ell_1} \to \mathbb{R}^{\ell_2}$, the KKT conditions are given as

$$\begin{array}{rl} \text{Find} & (x, y, z, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \\ \text{such that} & x \in \mathcal{K},\ y \in \mathcal{K},\ x^\top y = 0, \\ & y = G(z), \quad H(z) = 0, \\ & \nabla\theta(z) - \nabla G(z)x - \nabla H(z)w = 0, \end{array}$$

which is certainly of the form MSOCCP (3.1). Also there are some examples that cannot be written as MCP or NSOCP but as MSOCCP. Consider the non-cooperative game in which each player chooses his/her strategy by solving SOCP. Then the KKT conditions of each player's SOCP can be written as an MSOCCP, and all players' MSOCCPs are combined to one "big" MSOCCP. Such a formulation can be found in the robust Nash equilibrium problems [12, 17] and the robust Wardrop equilibrium problems [13].

For solving MSOCCP (3.1), we extend the regularized smoothing Newton algorithm (ReSNA) proposed by Hayashi, Yamashita and Fukushima [11]. Although the manner of extension is quite analogous, our study is different in the following two points:

(a) We focus on the *mixed* SOCCP;

(b) We analyze the convergence property under *Cartesian* P_0 assumption.

In terms of (a), Hayashi et al. [11] only focused on the non-mixed SOCCP:

$$\begin{aligned} \text{Find} \quad & (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \\ \text{such that} \quad & x \in \mathcal{K},\ y \in \mathcal{K},\ x^\top y = 0,\ y = F_1(x), \end{aligned} \tag{3.4}$$

which is a special case of MSOCCP (3.1) with $\ell = 0$. It is true that SOCP (3.3) can be cast as a non-mixed SOCCP (3.4) if some auxiliary variables are incorporated.[3] However, such a formulation is not always appropriate, since it increases the dimension of decision variables and may lose some favorable properties that the functions G and H in (3.3) possess. In terms of (b), the authors of [11] showed the convergence property under the monotonicity assumption. However, as mentioned later, the Cartesian P_0 property is strictly weaker than the monotonicity.

3.2 Regularized smoothing Newton method

Next, we construct the ReSNA for MSOCCP (3.1). Since the essential scheme is the same as [11], we will often omit the details.

Let x and y be partitioned according to the Cartesian structure of $\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_m}$, i.e.,

$$\begin{aligned} x &= (x^1, \ldots, x^m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}, \\ y &= (y^1, \ldots, y^m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}. \end{aligned} \tag{3.5}$$

Define function $\Phi_{\text{NR}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, called a *natural residual* [8, 11], by

$$\begin{aligned} \Phi_{\text{NR}}(x, y) &:= \begin{pmatrix} \varphi_{\text{NR}}(x^1, y^1) \\ \vdots \\ \varphi_{\text{NR}}(x^m, y^m) \end{pmatrix}, \\ \varphi_{\text{NR}}(x^i, y^i) &:= x^i - P_{\mathcal{K}^{n_i}}(x^i - y^i), \end{aligned}$$

where $P_{\mathcal{K}^{n_i}}(x^i - y^i)$ denotes the Euclidean projection of $x^i - y^i$ onto $\mathcal{K}^{n_i}$. Note that, when $n_i = 1$, we have $\varphi_{\text{NR}}(x^i, y^i) = \min(x^i, y^i)$ since $\mathcal{K}^1 = \mathbb{R}_+$ yields $x^i - P_{\mathcal{K}^1}(x^i - y^i) = x^i - \max(0, x^i - y^i) = \min(x^i, y^i)$. It is known that the natural residual Φ_{NR} satisfies

$$\Phi_{\text{NR}}(x, y) = 0 \iff x \in \mathcal{K},\ y \in \mathcal{K},\ x^\top y = 0.$$

[3] Let z' and z'' be auxiliary variables satisfying $z' \geq 0$, $z'' \geq 0$ and $z = z' - z''$. Then, SOCP (3.3) can be reformulated as another SOCP with decision variables $(z', z'') \in \mathbb{R}_+^{2\ell_1}$ and certain SOC constraints. The KKT conditions of such an SOCP can be expressed as a non-mixed SOCCP.

Therefore, letting $H_{\text{NR}} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^{2n+\ell}$ be

$$H_{\text{NR}}(x,y,p) := \begin{pmatrix} \Phi_{\text{NR}}(x,y) \\ F_1(x,p) - y \\ F_2(x,p) \end{pmatrix}, \tag{3.6}$$

we can reformulate MSOCCP (3.1) as the following VE equivalently:

$$H_{\text{NR}}(x,y,p) = 0. \tag{3.7}$$

Since MSOCCP (3.1) is equivalent to (3.7), we have only to solve (3.7) instead of MSOCCP (3.1). However, function Φ_{NR} is nondifferentiable, and hence the Newton based approach cannot be applied in a direct manner. Moreover, even if function Φ_{NR} is smoothened, its Jacobian matrix may become singular. To overcome those difficulties, we introduce the smoothing method and the regularization method.

Smoothing method

A function Φ_μ parameterized by $\mu \geq 0$ is called a *smoothing function* of Φ_{NR} if it satisfies the following conditions:

- For any fixed $\mu > 0$, Φ_μ is continuously differentiable over $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$;
- $\lim_{\mu \searrow 0} \Phi_\mu(x,y) = \Phi_{\text{NR}}(x,y)$ for any fixed $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

In the smoothing method, we handle Φ_μ instead of Φ_{NR} with letting $\mu \searrow 0$.

In [11], the smoothing function Φ_μ is composed as follows. Consider a continuously differentiable convex function $\hat{g} : \Re \to \Re$ satisfying

$$\lim_{\alpha \to -\infty} \hat{g}(\alpha) = 0, \ \ \lim_{\alpha \to \infty} (\hat{g}(\alpha) - \alpha) = 0, \ \ 0 < \hat{g}'(\alpha) < 1. \tag{3.8}$$

For example, $\hat{g}_1(\alpha) = (\sqrt{\alpha^2 + 4} + \alpha)/2$ and $\hat{g}_2(\alpha) = \ln(e^\alpha + 1)$ satisfy (3.8). Then, we can easily see that $\lim_{\mu \searrow 0} \mu \hat{g}(\alpha/\mu) = \max\{0, \alpha\}$ for any $\alpha \in \mathbb{R}$. By using this fact, Φ_μ is defined by

$$\Phi_\mu(x,y) := \begin{pmatrix} \varphi_\mu(x^1, y^1) \\ \vdots \\ \varphi_\mu(x^m, y^m) \end{pmatrix}$$

with

$$\begin{aligned} \varphi_\mu(x^i, y^i) &:= x^i - P_\mu(x^i - y^i), \\ P_\mu(z) &:= \begin{cases} \mu \hat{g}(\lambda_1/\mu) u^{\{1\}} + \mu \hat{g}(\lambda_2/\mu) u^{\{2\}} & (\dim(z) \geq 2) \\ \mu \hat{g}(z/\mu) & (\dim(z) = 1). \end{cases} \end{aligned} \tag{3.9}$$

Here, λ_i and $u^{\{i\}}$ $(i = 1, 2)$ denote the spectral values and vectors of z, respectively. Those definitions are according to the Euclidean Jordan algebra, and its details are found in [7, 8, 11].

Regularization method

Let the functions $F_{1,\varepsilon} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ and $F_{2,\varepsilon} : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^l$ be defined by

$$\begin{aligned} F_{1,\varepsilon}(x,p) &:= F_1(x,p) + \varepsilon x, \\ F_{2,\varepsilon}(x,p) &:= F_2(x,p) + \varepsilon p, \end{aligned}$$

respectively, with a positive parameter ε. In general, functions $F_{1,\varepsilon}$ and $F_{2,\varepsilon}$ have better properties than F_1 and F_2 from the viewpoint of global convergence. For example, if $F = \binom{F_1}{F_2}$ is a P_0 function, then $\binom{F_{1,\varepsilon}}{F_{2,\varepsilon}}$ is a uniformly P function for any $\varepsilon > 0$.

Embedding the smoothing and regularization parameters, we define a function $H_{\mu,\varepsilon} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{2n+\ell}$ by

$$H_{\mu,\varepsilon}(x,y,p) := \begin{pmatrix} \Phi_\mu(x,y) \\ F_{1,\varepsilon}(x,p) - y \\ F_{2,\varepsilon}(x,p) \end{pmatrix}. \tag{3.10}$$

Then, we solve the inequality $\|H_{\mu,\varepsilon}(x,y,p)\| \leq \beta$ by Newton's method with letting $(\mu,\varepsilon,\beta) \searrow (0,0,0)$. This is the main idea of ReSNA.

Before providing the main algorithm, we give some functions and its related proposition that will be used in the algorithm. Since the functions are important only for the local quadratic convergence,[4] we omit the detailed explanations here.

Definition 3.2 [11]

(a) $\tilde{\lambda} : \mathbb{R}^n \to [0,+\infty)$ *is a function defined by*

$$\tilde{\lambda}(z) := \begin{cases} \min\limits_{i \in \mathcal{I}(z)} |\lambda_i(z)| & (\mathcal{I}(z) \neq \emptyset) \\ 0 & (\mathcal{I}(z) = \emptyset), \end{cases} \tag{3.11}$$

where $\lambda_i(z)$ $(i = 1,2)$ *are the spectral values of* z, *and* $\mathcal{I}(z) \subseteq \{1,2\}$ *is the index set defined by* $\mathcal{I}(z) := \{i \mid \lambda_i(z) \neq 0\}$.

(b) *Choose any function* $\hat{g}$ *satisfying* (3.8). *Then,* $\overline{\mu} : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ *is an arbitrary function such that*

$$\left| \hat{g}'(\alpha/\mu) - \lim_{\mu \searrow 0} \hat{g}'(\alpha/\mu) \right| < \delta \qquad \forall \mu \in \big(0,\, \overline{\mu}(\alpha,\delta)\big), \tag{3.12}$$

for any fixed $\alpha \in \mathbb{R}$ *and* $\delta > 0$.

Proposition 3.1 [11, Prop. 4.12] *Let* $\hat{g}$ *be defined by* $\hat{g}(\alpha) = (\sqrt{\alpha^2 + 4} + \alpha)/2$, *which satisfies* (3.8). *Let* $\overline{\mu} : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ *be defined by*

$$\overline{\mu}(\alpha,\delta) := \begin{cases} +\infty & (\delta \geq 1/2 \ \text{ or } \ \alpha = 0) \\ \dfrac{1}{2}|\alpha|\sqrt{\delta} & (\delta < 1/2 \ \text{ and } \ \alpha \neq 0). \end{cases}$$

Then, $\overline{\mu}$ *satisfies the condition* (3.12).

[4] In [11], the authors showed not only the global convergence but also the local quadratic convergence. However, we do not discuss the latter property since the ISIA prototype cannot affect the quadratic convergence structurally.

Now, we are in the position to provide the detailed steps of ReSNA. In what follows, we use the following notations for convenience:

$$w := \begin{pmatrix} x \\ y \\ p \end{pmatrix}, \quad w^{(k)} := \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ p^{(k)} \end{pmatrix}.$$

Algorithm 2 (Regularized smoothing Newton algorithm: ReSNA)

Step 0 *Choose the parameters* η, $\rho \in (0,1)$, $\overline{\eta} \in (0,\eta]$, $\sigma \in (0,1/2)$, $\kappa > 0$ *and* $\hat{\kappa} > 0$. *Choose the initial values* $w^{(0)} \in \mathbb{R}^{2n+\ell}$ *and* $\beta_0 \in (0,\infty)$. *Let* $\mu_0 := \|H_{\text{NR}}(w^{(0)})\|$ *and* $\varepsilon_0 := \|H_{\text{NR}}(w^{(0)})\|$. *Set* $k := 0$.

Step 1 *Terminate if* $\|H_{\text{NR}}(w^{(k)})\| = 0$.

Step 2

Step 2.0 *Set* $v^{(0)} := w^{(k)} \in \mathbb{R}^{2n+\ell}$ *and* $j := 0$.

Step 2.1 *Find a vector* $\hat{d}^{(j)} \in \mathbb{R}^{2n+\ell}$ *such that*

$$H_{\mu_k,\varepsilon_k}(v^{(j)}) + \nabla H_{\mu_k,\varepsilon_k}(v^{(j)})^\top \hat{d}^{(j)} = 0.$$

Step 2.2 *If* $\|H_{\mu_k,\varepsilon_k}(v^{(j)} + \hat{d}^{(j)})\| \le \beta_k$, *then let* $w^{(k+1)} := v^{(j)} + \hat{d}^{(j)}$ *and go to* Step 3. *Otherwise, go to* Step 2.3.

Step 2.3 *Find the smallest nonnegative integer* m *such that*

$$\|H_{\mu_k,\varepsilon_k}(v^{(j)} + \rho^m \hat{d}^{(j)})\|^2 \le (1 - 2\sigma\rho^m)\|H_{\mu_k,\varepsilon_k}(v^{(j)})\|^2.$$

Let $m_j := m$, $\tau_j := \rho^{m_j}$ *and* $v^{(j+1)} := v^{(j)} + \tau_j \hat{d}^{(j)}$.

Step 2.4 *If*

$$\|H_{\mu_k,\varepsilon_k}(v^{(j+1)})\| \le \beta_k, \tag{3.13}$$

then let $w^{(k+1)} := v^{(j+1)}$ *and go to* Step 3. *Otherwise, set* $j := j+1$ *and go back to* Step 2.1.

Step 3 *Update the parameters as follows*:

$$\begin{aligned}
\mu_{k+1} &:= \min\left\{\kappa\|H_{\text{NR}}(w^{(k+1)})\|^2,\, \mu_0\overline{\eta}^{k+1},\, \overline{\mu}\Big(\tilde{\lambda}(x^{(k+1)} - y^{(k+1)}), \hat{\kappa}\|H_{\text{NR}}(w^{(k+1)})\|\Big)\right\}, \\
\varepsilon_{k+1} &:= \min\left\{\kappa\|H_{\text{NR}}(w^{(k+1)})\|^2, \varepsilon_0\overline{\eta}^{k+1}\right\}, \\
\beta_{k+1} &:= \beta_0\eta^{k+1}.
\end{aligned}$$

Set $k := k+1$. *Go back to* Step 1.

Steps 2.0 − 2.4 are to find a point $w^{(k+1)}$ such that $\|H_{\mu_k,\varepsilon_k}(w^{(k+1)})\| \le \beta_k$. We note that Algorithm 2 is well-defined in the sense that Steps 2.0 − 2.4 find $v^{(j+1)}$ satisfying (3.13) in a finite number of iterations for each k. (It can be proved easily as in [11].) In Step 3, $\tilde{\lambda}$ and $\overline{\mu}$ are defined as (3.11) and (3.12), respectively. This step specifies the updating rule of the parameters, where $\{\beta_k\}$, $\{\mu_k\}$ and $\{\varepsilon_k\}$ converge to 0 since $0 < \overline{\eta} \le \eta < 1$.

4 Convergence analyses under Cartesian P_0 property

In the previous section, we have provided the detailed steps of ReSNA for MSOCCP (3.1). In the algorithm, the MSOCCP is reformulated as VE (3.7) equivalently. Moreover, to solve it, the ReSNA generates the sequence $\{w^{(k)}\}$ such that (i) $\|H_{\mu_k,\varepsilon_k}(w^{(k+1)})\| \le \beta_k$ for each k, (ii) $\{\beta_k\}$ converges to 0, and (iii) $\{H_{\mu_k,\varepsilon_k}\}$ converges to $H_{\rm NR}$ uniformly. Thus, if $H_{\rm NR}$ is weakly univalent, then we can apply the ISIA prototype directly to prove the global convergence.

In this section, we first introduce the Cartesian P_0 property to a certain function in MSOCCP (3.1). Then, we prove that $H_{\mu,\varepsilon}$ is injective for any $\mu > 0$ and $\varepsilon > 0$, and consequently, Algorithm 2 is globally convergent.

4.1 Cartesian P_0 property

Let

$$\sigma := (\nu_1, \nu_2, \dots, \nu_r)^\top \in \mathbb{Z}^r \tag{4.1}$$

be an integer vector such that $\nu_i \ge 1$ for $i = 1, 2, \dots, r$ and $\nu = \sum_{i=1}^r \nu_i$. Then, we first decompose the vector $z \in \mathbb{R}^\nu$, matrix $M \in \mathbb{R}^{\nu\times\nu}$ and function $F : \mathbb{R}^\nu \to \mathbb{R}^\nu$ according to the components of σ as follows:

$$z = \begin{bmatrix} z^1 \\ z^2 \\ \vdots \\ z^r \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1r} \\ M_{21} & M_{22} & \cdots & M_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rr} \end{bmatrix}, \quad F(z) = \begin{bmatrix} F^1(z) \\ F^2(z) \\ \vdots \\ F^r(z) \end{bmatrix},$$

where $z^i \in \mathbb{R}^{\nu_i}$, $M_{ij} \in \mathbb{R}^{\nu_i\times\nu_j}$ and $F^i : \mathbb{R}^\nu \to \mathbb{R}^{\nu_i}$. Then, we can define the Cartesian P_0 property.

Definition 4.3 *Let $\sigma \in \mathbb{Z}^r$ be an integer vector given as* (4.1)*. Then, we say that*

(i) *the matrix $M \in \mathbb{R}^{\nu\times\nu}$ satisfies the σ-Cartesian P_0 property if, for any $z \in \mathbb{R}^\nu \setminus \{0\}$, there exists $i = i(z) \in \{1, 2, \dots, r\}$ such that*

$$(z^i)^\top (Mz)^i \ge 0 \quad \text{and} \quad z^i \ne 0;$$

(ii) *the function $F : \mathbb{R}^\nu \to \mathbb{R}^\nu$ satisfies the σ-Cartesian P_0 property if, for any $(x, y) \in \mathbb{R}^\nu \times \mathbb{R}^\nu$, there exists $i = i(x, y) \in \{1, 2, \dots, r\}$ such that*

$$(x^i - y^i)^\top (F^i(x) - F^i(y)) \ge 0 \quad \text{and} \quad x^i \ne y^i.$$

Remark If $\sigma = (1, 1, \dots, 1)^\top \in \mathbb{R}^\nu$ and $r = \nu$, then the σ-Cartesian P_0 property coincides with the normal P_0 property. On the other hand, if $\sigma = \nu \in \mathbb{R}$ and $r = 1$, then the σ-Cartesian P_0 properties of (i) and (ii) are equivalent to the positive semidefiniteness and the monotonicity, respectively. Cartesian P property and uniform Cartesian P property can be defined in a similar way to the case of normal P property and uniform P property.

It is known that, if M is a P_0 matrix, then $M + D$ is nonsingular for any positive definite diagonal matrix D. The following theorem is a natural extension of this property and provides the necessary condition for a given matrix to have the Cartesian P_0 property.

Theorem 4.2 *Let σ be given by (4.1), and $M \in \mathbb{R}^{\nu\times\nu}$ be an arbitrary σ-Cartesian P_0 matrix. Then, $M + D$ is nonsingular for any positive definite block diagonal matrix $D = \mathrm{diag}\{D_{ii}\}_{i=1}^r$ with $D_{ii} \in \mathbb{R}^{\nu_i\times\nu_i} \succ 0$.*

proof. Let z be a vector such that $(M + D)z = 0$. Assume for contradiction that $z \neq 0$. Then, due to the σ-Cartesian P_0 property, there exists an i such that $z^i \neq 0$ and $(z^i)^\top(Mz)^i \geq 0$. Thus we have

$$0 = (z^i)^\top\big((M + D)z\big)^i = (z^i)^\top(Mz)^i + (z^i)^\top D_{ii}z^i \geq (z^i)^\top D_{ii}z^i,$$

where the first equality is due to $(M + D)z = 0$. However, this contradicts the positive definiteness of D_{ii} and $z^i \neq 0$. Thus $M + D$ is nonsingular. ■

4.2 Global convergence analysis

We first analyze the nonsingularity of the Jacobian matrix of $H_{\mu,\varepsilon}(x, y, p)$ defined by (3.10). Let $\mu > 0$ and $\varepsilon \geq 0$ be given arbitrarily. Then, by the definition of $H_{\mu,\varepsilon}$, Φ_μ, $\hat{g}$, etc., the Jacobian $\nabla H_{\mu,\varepsilon}(x, y, p) \in \mathbb{R}^{(2n+\ell)\times(2n+\ell)}$ can be calculated as

$$\nabla H_{\mu,\varepsilon}(x, y, p) = \begin{pmatrix} I - D_\mu(x, y) & \nabla_x F_1(x, p) + \varepsilon I & \nabla_x F_2(x, p) \\ D_\mu(x, y) & -I & 0 \\ 0 & \nabla_p F_1(x, p) & \nabla_p F_2(x, p) + \varepsilon I \end{pmatrix}, \tag{4.2}$$

where

$$D_\mu(x, y) := \mathrm{diag}\left\{\nabla P_\mu(x^i - y^i)\right\}_{i=1}^m. \tag{4.3}$$

In (4.3), P_μ is defined by (3.9), and $\mathrm{diag}\left\{\nabla P_\mu(x^i - y^i)\right\}_{i=1}^m$ denotes the block diagonal matrix with entries $\nabla P_\mu(x^i - y^i) \in \mathbb{R}^{n_i\times n_i}$ $(i = 1, \ldots, m)$. The explicit expression of $\nabla P_\mu(\cdot)$ is given in [11]. For this Jacobian function, we have the following property.

Proposition 4.2 [8] *Let $P_\mu : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ be defined by (3.9). Then we have*

$$0 \prec \nabla P_\mu(z) \prec I \tag{4.4}$$

for any $z \in \mathbb{R}^{n_i}$, where $A \prec B$ means the positive definiteness of $B - A$.

Notice that, if $n_i = 1$, then (4.4) means that $0 < P'_\mu(z) < 1$.

By using this fact, we can prove the nonsingularity of $\nabla H_{\mu,\varepsilon}$ under Cartesian P_0 assumption. Set $\sigma \in \mathbb{Z}^{m+\ell}$ as

$$\sigma := (n_1, n_2, \ldots, n_m, 1, \ldots, 1)^\top \in \mathbb{Z}^{m+\ell}, \tag{4.5}$$

where the first m components $n_1, n_2, \ldots, n_m$ corresponds to the dimensions of SOCs $\mathcal{K}$, and the last ℓ components $1, \ldots, 1$ corresponds to the ℓ-dimensional vector equality $F_2(x, p) = 0$. Moreover, let $F : \mathbb{R}^{n+\ell} \to \mathbb{R}^{n+\ell}$ be defined by

$$F(x, p) := \begin{pmatrix} F_1(x, p) \\ F_2(x, p) \end{pmatrix}. \tag{4.6}$$

Then, we have the following theorem.

Theorem 4.3 *Let $\sigma \in \mathbb{Z}^{m+\ell}$ and $F : \mathbb{R}^{n+\ell} \to \mathbb{R}^{n+\ell}$ be given by* (4.5) *and* (4.6)*, respectively. Suppose that F is a σ-Cartesian P_0 function. Then,* (a) *the matrix $\nabla H_{\mu,\varepsilon}(x, y, p)$ given by* (4.2) *is nonsingular for any $\mu > 0$, $\varepsilon > 0$ and $(x, y, p) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell$, and hence,* (b) *the function H_{NR} defined by* (3.6) *is weakly univalent.*

proof. Let $\xi := (\xi_x, \xi_y, \xi_p) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell$ be a vector satisfying $\nabla H_{\mu,\varepsilon}(x, y, p)\xi = 0$. Denote $D_\mu = D_\mu(x, y)$, $F_1 = F_1(x, y, p)$ and $F_2 = F_2(x, y, p)$ for simplicity. Then, by (4.2), we have

$$(I - D_\mu)\xi_x + (\nabla_x F_1 + \varepsilon I)\xi_y + \nabla_x F_2 \xi_p = 0, \tag{4.7}$$

$$D_\mu \xi_x - \xi_y = 0, \tag{4.8}$$

$$\nabla_p F_1 \xi_y + (\nabla_p F_2 + \varepsilon I)\xi_p = 0. \tag{4.9}$$

By (4.8) together with (4.3) and (4.4), we have $\xi_x = D_\mu^{-1}\xi_y$. Substituting this into (4.7) and (4.9), we have $(D_\mu^{-1} - I + \nabla_x F_1 + \varepsilon I)\xi_y + \nabla_x F_2 \xi_p = 0$ and $\nabla_p F_1 \xi_y + (\nabla_p F_2 + \varepsilon I)\xi_p = 0$, that is,

$$\begin{aligned} 0 &= \left(\begin{bmatrix} \nabla_x F_1 & \nabla_x F_2 \\ \nabla_p F_1 & \nabla_p F_2 \end{bmatrix} + \begin{bmatrix} D_\mu^{-1} - I + \varepsilon I & 0 \\ 0 & \varepsilon I \end{bmatrix} \right) \begin{bmatrix} \xi_y \\ \xi_p \end{bmatrix} \\ &= \left(\nabla F + \begin{bmatrix} \operatorname{diag}\left\{ \nabla P_\mu(x^i - y^i)^{-1} - I \right\}_{i=1}^m + \varepsilon I & 0 \\ 0 & \varepsilon I \end{bmatrix} \right) \begin{bmatrix} \xi_y \\ \xi_p \end{bmatrix}. \end{aligned} \tag{4.10}$$

Notice that $\nabla P_\mu(x^i - y^i)^{-1} - I \succ 0$ since $0 \prec \nabla P_\mu(x^i - y^i) \prec I$. Moreover, $\nabla F(x, p)$ is a σ-Cartesian P_0 matrix since F is a σ-Cartesian P_0 function. Hence, by Theorem 4.2, the matrix

$$\nabla F + \begin{bmatrix} \operatorname{diag}\left\{ \nabla P_\mu(x^i - y^i)^{-1} - I \right\}_{i=1}^m + \varepsilon I & 0 \\ 0 & \varepsilon I \end{bmatrix}$$

is nonsingular. This together with (4.10) yields $\xi_y = 0$, $\xi_p = 0$ and $\xi_x = D_\mu^{-1}\xi_y = 0$. Hence $\nabla H_{\mu,\varepsilon}(x, y, p)$ is nonsingular. ∎

Finally, we show the global convergence of Algorithm 2.

Theorem 4.4 *Let $\sigma \in \mathbb{Z}^{m+\ell}$ and $F : \mathbb{R}^{n+\ell} \to \mathbb{R}^{n+\ell}$ be given by* (4.5) *and* (4.6)*, respectively. Suppose that* (i) *the solution set of MSOCCP* (3.1) *is nonempty and bounded, and* (ii) *F is a σ-Cartesian P_0 function. Then, the sequence $\{w^k\}$ generated by Algorithm* 2 *is bounded, and any accumulation point solves MSOCCP* (3.1)*.*

proof. By the definition (3.10) of $H_{\mu,\varepsilon}$ and Theorem 4.3, we can easily see that the assumptions (i)–(iii) of Corollary 2.1 holds. Moreover, by the same argument in [11], we have $v^{(j+1)}$ satisfying (3.13) with a finite j, i.e., Assumption B holds. Hence, by Corollary 2.1, we obtain the result. ∎

5 Concluding remarks

In this paper, we have mainly dealt with two topics. The first one is the ISIA for solving weakly univalent VEs. We proved that, under the boundedness of the solution set, the sequence generated by the ISIA is bounded and any accumulation point solves the VE. The second topic is the ReSNA for MSOCCP. Applying the ISIA prototype to the ReSNA, we have showed the boundedness and the global convergence of generated sequence under the assumption that a certain function in MSOCCP has the Cartesian P_0 property.

We emphasize again that the ISIA plays a role of essential and comprehensive prototype not only for the ReSNA but also for many other algorithms, which may be applied to conic complementarity problems, nonlinear optimization problems, semi-infinite optimization problems, variational inequality problems, etc. Some papers on Newton type algorithms only shows that the accumulation point of generated sequence is the solution *if the generated sequence is bounded*. However, if they meet the ISIA prototype, they guarantee the boundedness of the generated sequence, too. For the ReSNA, another important future issue is to show the superlinear or quadratic convergence. To this end, the ISIA prototype is not sufficient, but we have to analyze other properties such as the Jacobian consistency and the semismoothness [11]. Another possible future issue is to extend the ReSNA to the mixed symmetric cone complementarity problems. The ReSNA for the non-mixed version was already proposed in [14] and the convergence results were obtained in a way similar to [11]. Therefore, we can expect that the global convergence analyses by the ISIA prototype is possible to the mixed version, too.

References

[1] S. Bi, S. Pan, and J.-S. Chen, *The same growth of FB and NR symmetric cone complementarity functions*, Optimization Letters, 6 (2012), pp. 153–162.

[2] J.-S. Chen, *A new merit function and its related properties for the second-order cone complementarity problem*, Pacific Journal of Optimization, 2 (2006), pp. 167–179.

[3] J.-S. Chen and S. Pan, *A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs*, Pacific Journal of Optimization, 8 (2012), pp. 33–74.

[4] J.-S. Chen and P. Tseng, *An unconstrained smooth minimization reformulation of the second-order cone complementarity problem*, Mathematical Programming, 104 (2005), pp. 293–327.

[5] S. Chen, L.-P. Pang, and D. Li, *An inexact semismooth newton method for variational inequality with symmetric cone constraints*, Journal of Industrial and Management Optimization, 11 (2015), pp. 733–746.

[6] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer-Verlag, New York, 2003.

[7] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, New York, 1994.

[8] M. Fukushima, Z.-Q. Luo, and P. Tseng, *Smoothing functions for second-order cone complementarity problems*, SIAM Journal on Optimization, 12 (2001), pp. 436–460.

[9] M. S. Gowda and R. Sznajder, *Weak univalence and connectedness of inverse images of continuous functions*, Mathematics of Operations Research, 24 (1999), pp. 255–261.

[10] P. T. Harker and J.-S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications*, Mathematical Programming, 48 (1990), pp. 161–220.

[11] S. Hayashi, N. Yamashita, and M. Fukushima, *A combined smoothing and regularization method for monotone second-order cone complementarity problems*, SIAM Journal on Optimization, 15 (2005), pp. 593–615.

[12] ——, *Robust Nash equilibria and second-order cone complementarity problems*, Journal of Nonlinear and Convex Analysis, 6 (2005), pp. 283–296.

[13] Y. Ito, *Robust wardrop equilibria in the traffic assignment problem with uncertain data*, Master's thesis, Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, 2011.

[14] L. Kong, J. Sun, and N. Xiu, *A regularized smoothing Newton method for symmetric cone complementarity problems*, SIAM Journal on Optimization, 19 (2008), pp. 1028–1047.

[15] Y. Narushima, H. Ogasawara, and S. Hayashi, *A smoothing method with appropriate parameter control based on Fischer-Burmeister function for second-order cone complementarity problems*, Abstract and Applied Analysis, 2013 (2013). 16 pages.

[16] Y. Narushima, N. Sagara, and H. Ogasawara, *A smoothing Newton method with Fischer-Burmeister function for second-order cone complementarity problems*, Journal of Optimization Theory and Applications, 149 (2011), pp. 79–101.

[17] R. Nishimura, S. Hayashi, and M. Fukushima, *Robust Nash equilibria in N-person non-cooperative games: Uniqueness and reformulation*, Pacific Journal of Optimization, 5 (2009), pp. 237–259.

[18] L. Qi and D. Sun, *Smoothing functions and smoothing newton method for complementarity and variational inequality problems*, Journal of Optimization Theory and Applications, 113 (2002).

[19] J. Zhang and K. Zhang, *An inexact smoothing method for the monotone complementarity problem over symmetric cones*, Optimization Methods and Software, 27 (2012), pp. 445–459.